Beta and Gamma Functions

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Outline

1. Improper Integrals
   - The Gamma Function
   - The Beta Function
   - Properties
Improper Integrals

- Improper integrals are definite integrals that cover an unbounded area. For example, \( \int_{1}^{\infty} \frac{1}{x^2} \, dx \)

- An **improper integral** is a definite integral that has either one or both limits infinite or an integrand that approaches infinity at one or more points in the range of integration.

- Hence improper integrals are of **two type**.
1 Type-1: (Infinite limits of integration): In this kind of integrals one or both limits of integration are infinity. For example, \( \int_{1}^{\infty} \frac{1}{x^2} dx \) is an improper integral.

It can be viewed as the limit \( \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x^2} dx \).

2 Type-2: (Discontinuous integrand or integrands with vertical asymptotes): In this type of improper integrals the endpoints are finite, but the integrand function is unbounded at one (or two) of the endpoints. For example, \( \int_{0}^{1} \frac{1}{\sqrt{x}} dx \).

It can be viewed as the limit \( \lim_{a \to 0^+} \int_{a}^{1} \frac{1}{\sqrt{x}} dx \).
Remark

- *Not all improper integrals have a finite value, but some of them definitely do.*

- *When the limit exists we say the integral is convergent, and when it doesn’t we say it’s divergent.*
The Gamma Function

- The **Gamma function** is important as it is an extension to the factorial function $f(n) = n!$ for all $n \in \mathbb{N}$.

- The **Gamma function** is defined as the single variable function

  $$
  \Gamma(x) = \int_{0}^{\infty} e^{-t} t^{x-1} \, dt, \quad x > 0.\tag{1}
  $$

- By using integration by parts we find that

  $$
  \Gamma(x + 1) = \int_{0}^{\infty} e^{-t} t^{x} \, dt = x \int_{0}^{\infty} e^{-t} t^{x-1} \, dt = x \Gamma(x).
  $$

  and 

  $$
  \Gamma(x + 1) = x!, \quad \text{if } x \in \mathbb{Z}^+.
  $$
From $\Gamma(x + 1) = x\Gamma(x)$, it is clear that if $\Gamma(x)$ is known throughout a unit interval say: $1 \leq x \leq 2$, then the value of $\Gamma(x)$ throughout the next unit interval $2 < x \leq 3$ are found and so on. In this way, the values of $\Gamma(x)$ for all positive values of $x > 1$ may be found.

From $\Gamma(x) = \frac{\Gamma(x + 1)}{x}$, it is clear that if $\Gamma(x)$ is known throughout a unit interval say: $1 < x \leq 2$, then the value of $\Gamma(x)$ throughout the previous unit interval $0 < x \leq 1$ are found and so on.
**Note:** From $\Gamma(x + 1) = x\Gamma(x)$ and $\Gamma(x) = \frac{\Gamma(x + 1)}{x}$, it is clear that $\Gamma(x)$ is exists for all values of $x$ except when $x = 0$ or a negative integer.
The Beta Function

- The Beta Function is defined as the two variable function

\[ B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} \, dt, \quad \text{for } x, y > 0. \] (2)

- \[ B(x, y) = 2 \int_0^{\pi/2} (\sin \theta)^{2x-1} (\cos \theta)^{2y-1} \, d\theta, \quad \text{put } t = \sin^2 \theta. \]

- \[ B(x, y) = \int_0^{\infty} \frac{u^{x-1}}{(1 + u)^{x+y}} \, du, \quad \text{put } t = \frac{1}{1 + u}. \]

- From the definition it is easily seen that \( B(x, y) = B(y, x). \)
Example

Evaluate \( \int_{0}^{1} \frac{dx}{\sqrt{1 - x^4}} \).

Solution

Substitute \( x^2 = \sin \theta \), then we obtain

\[
\int_{0}^{1} \frac{dx}{\sqrt{1 - x^4}} = \frac{1}{2} \int_{0}^{\pi/2} (\sin \theta)^{-1/2} d\theta
\]

using the definition of beta function (polar form), we get

\[
= \frac{\sqrt{\pi}}{4} \frac{\Gamma(1/4)}{\Gamma(3/4)}
\]
Example

Evaluate \( \int_{0}^{\infty} \frac{x^c}{c^x} \, dx \).

Solution

\[
\int_{0}^{\infty} \frac{x^c}{c^x} \, dx = \int_{0}^{\infty} e^{-x \log c} x^c \, dx
\]
\[
= \int_{0}^{\infty} e^{-t} \left( \frac{t}{\log c} \right)^c \frac{dt}{\log c}
\]
\[
= \frac{1}{(\log c)^{c+1}} \int_{0}^{\infty} e^{-t} t^c \, dt
\]
\[
= \frac{1}{(\log c)^{c+1}} \Gamma(c + 1) \quad \text{(by the Gamma function definition)}
\]
Example

Evaluate \( \int_0^\infty e^{-ax} x^{m-1} \sin bx \, dx \).

Solution

By substituting \( x = ay \) in \( \Gamma(m) = \int_0^\infty e^{-x} x^{m-1} \, dx \), we obtain
\[
\int_0^\infty e^{-ay} y^{m-1} \, dy = \frac{\Gamma(m)}{a^m}.
\]
Thus
\[
\int_0^\infty e^{-ax} x^{m-1} \sin bx \, dx = IP \int_0^\infty e^{-ax} x^{m-1} e^{ibx} \, dx \quad (\because \sin bx = IP(e^{ibx}))
\]
\[
= IP \int_0^\infty e^{-(a-ib)x} x^{m-1} \, dx = IP \frac{\Gamma(m)}{(a - ib)^m}
\]
using \( a = r \cos \theta, \ b = r \sin \theta, \ r^2 = a^2 + b^2 \)
\[
= \frac{\Gamma(m)}{(a^2 + b^2)^{m/2}} \sin m\theta.
\]
Theorem (Relation between beta and gamma functions)

The connection between the beta function and the gamma function is given by

\[ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}. \]

In order to prove, we use the definition (1) to obtain

\[ \Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1}e^{-t}dt \int_0^\infty s^{y-1}e^{-s}ds = \int_0^\infty \int_0^\infty e^{-(t+s)} t^{x-1}s^{y-1}dt \, ds. \]
Now we apply change of variables \( t = uv \) and \( s = u(1 - v) \) to this double integral.

Note that \( t + s = u \) and that \( 0 < t < \infty \) and \( 0 < s < \infty \) imply that \( 0 < u < \infty \) and \( 0 < v < 1 \).

The Jacobian of this transformation is

\[
\frac{\partial (t,s)}{\partial (u,v)} = \begin{vmatrix} v & u \\ 1 - v & -u \end{vmatrix} = -u.
\]

Therefore,

\[
dt \, ds = \left| \frac{\partial (t,s)}{\partial (u,v)} \right| \, du \, dv = u \, du \, dv.
\]
Hence, we have

\[ \Gamma(x)\Gamma(y) = \int_{v=0}^{1} \int_{u=0}^{\infty} e^{-u} (u \, v)^{x-1} (u(1 - v))^{y-1} \, u \, du \, dv \]

\[ = \int_{u=0}^{\infty} e^{-u} u^{x+y-1} \, du \int_{v=0}^{1} v^{x-1} (1 - v)^{y-1} \, dv \]

\[ = \Gamma(x + y)B(x, y) \]

Dividing both sides by \( \Gamma(x + y) \) gives the desired result.
Let us see the application of the previous Theorem.

Example

Prove that $\Gamma(1/2) = \sqrt{\pi}$.

Solution

*Since* $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$, *if* $x = y = 1/2$, *then we have*

$$B(1/2, 1/2) = \{\Gamma(1/2)\}^2, \quad \therefore \Gamma(1) = 1.$$

*Now, using the definition of beta function (polar form), we have*

$$B(1/2, 1/2) = \pi \implies \Gamma(1/2) = \sqrt{\pi}$$
Example

Given that \( \int_0^\infty \frac{x^{n-1}}{1 + x} \, dx = \frac{\pi}{\sin(n\pi)} \), then show that

\[
\Gamma(n)\Gamma(1 - n) = \frac{\pi}{\sin(n\pi)}.
\]

Solution

We know that \( B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)} = \int_0^\infty \frac{x^{n-1}}{(1 + x)^{m+n}} \, dx \).

Taking \( m + n = 1 \) or \( m = 1 - n \) and using given result, we obtain the required result.
Example

Prove that \( B(m, n) = B(m + 1, n) + B(m, n + 1) \).

Solution

*By the definition of the beta function, we have*

\[
B(m + 1, n) = 2 \int_0^{\pi/2} (\sin \theta)^{2m+1} (\cos \theta)^{2n-1} d\theta \\
= 2 \int_0^{\pi/2} (\sin \theta)^{2m-1} (\cos \theta)^{2n-1} (\sin \theta)^2 d\theta \\
= 2 \int_0^{\pi/2} (\sin \theta)^{2m-1} (\cos \theta)^{2n-1} d\theta - 2 \int_0^{\pi/2} (\sin \theta)^{2m-1} (\cos \theta)^{2n+1} d\theta \\
= B(m, n) - B(m, n + 1).
\]
Example (duplication formula)

Prove that $\Gamma(n)\Gamma(n + 1/2) = 2^{1-2n}\sqrt{\pi}\Gamma(2n)$.

Solution

By the definition of beta function, we have

$$B(n, n) = 2 \int_0^{\pi/2} (\sin \theta)^{2n-1}(\cos \theta)^{2n-1} d\theta$$

$$= 2.2^{1-2n} \int_0^{\pi/2} (\sin 2\theta)^{2n-1} d\theta \quad \text{(put} \ 2\theta = \phi)$$

$$= 2^{1-2n} \int_0^{\pi} (\sin \phi)^{2n-1} d\phi = 2^{1-2n}.2 \int_0^{\pi/2} (\sin \phi)^{2n-1} d\phi$$

$$\frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} = 2^{1-2n}.B(n, 1/2) = 2^{1-2n}\frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n + 1/2)}$$

after simplification, we obtain the required result.
For the video lecture use the following link

https://youtube.com/channel/UCK9ICMqdkO0GREITx-2UaEw

THANK YOU